

Multiplicative McKay correspondence in the symplectic case

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Introduction

McKay correspondence as a subject appeared approximately twenty-five years ago; for an excellent overview we refer the reader to M. Reid's Bourbaki talk [R2]. The basic setup is the following. Let V be a finite-dimensional complex vector space, and let $G \subset SL(V)$ be a finite subgroup which acts on V preserving the volume. The quotient $Y = V/G$ is a singular algebraic variety with trivial canonical bundle. Assume given a smooth projective resolution $X \rightarrow Y$ which is *crepant* — in other words, such that the canonical bundle K_X is trivial. What can one say about the topological invariants of

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X , starting from the combinatorics of the G -action on V ? Somewhat surprisingly, it turns out that one can actually say quite a lot. This is, roughly speaking, what is called the McKay correspondence.

For some time, all the research concentrated on the case of $\dim V = 2$ (in particular, this was the case with the original paper [McK] of J. McKay which gave name to the whole subject). An important breakthrough was achieved in [R1]. Drawing on the case of $\dim V = 2$, well-understood by that time, and also on some physics papers and on previous work in $\dim 3$, M. Reid has proposed several conjectures for vector spaces of arbitrary dimension. Essentially, he predicted what the homology $H_*(X, \mathbb{Q})$, the cohomology $H^*(X, \mathbb{Q})$ and the K -theory $K_0(X)$ should be, and gave some pointers to possible proofs.

There has been considerable progress since then, and it mostly concentrated on two areas. Firstly, most of the conjectures in [R1] have been proved ([Bat], [DL]) by applying M. Kontsevich's method of *motivic integration*. Secondly, W. Chen and Y. Ruan [CR] have introduced a new invariant for the pair $\langle V, G \subset SL(V) \rangle$ called the *orbifold cohomology algebra* $H_{orb}([V/G])$. As a vector space, this ring coincides with what McKay correspondence predict must be $H^*(X)$. However, and this is the new ingredient, it also carries a structure of a commutative algebra. A natural question, then, would be to take a crepant resolution $X \rightarrow V/G$ and to compare the multiplication $H_{orb}^*([V/G])$ with the standard multiplication in $H^*(X)$.

Unfortunately, the only general technique in McKay correspondence, — that of motivic integration, — does not seem to apply to this problem. And on the other hand, in the general case it is not even expected that the two multiplicative structures in $H_{orb}^*([V/G]) \cong H^*(X)$ are the same (see [Ru2] for precise conjectures). In order to get results, one has to restrict generality in some way.

The particular case that we will consider in this paper is that of *symplectic* vector spaces — we will assume that the vector space V is equipped with a non-degenerate skew-symmetric form, and that the finite group $G \subset Sp(V) \subset SL(V)$ preserves not only the volume, but also the symplectic form.

In this case, there are several simplifications. Firstly, the additive McKay correspondence can be established explicitly, without recourse to the motivic integration techniques ([Ka2]). One also gets a basis in cohomology given by classes of explicit algebraic cycles. Secondly, the orbifold cohomology space and the orbifold cohomology algebra are much easier to describe than

in the general case. However, the most important simplification is that the multiplication in $H^\bullet(X, \mathbb{Q})$ is known at least in one particular example — one takes $V = \mathbb{C}^{2n}$, one takes G to be the symmetric group on n letters, and one takes $X \rightarrow V/G$ to be the Hilbert scheme of subschemes in \mathbb{C}^2 of dimension 0 and length n . In fact, the multiplicative structure in $H^\bullet(X, \mathbb{Q})$ has been discovered independently in [LS] and [V] (see also [LQW]) without any reference to the general orbifold cohomology construction of [CR]. It is only later that it was realized that the cohomology algebras are one and the same. Y. Ruan [Ru1] has conjectured that this should always be true in the symplectic case, no matter what particular crepant resolution one takes.

The goal of this paper is to present a result proved recently by the author together with V. Ginzburg. The result, which is [GK, Theorem 1.2], establishes the Ruan conjecture: we prove that in the symplectic case, the cohomology $H^\bullet(X, \mathbb{C})$ of *any* crepant resolution $X \rightarrow V/G$ coincides with the Chen-Ruan orbifold cohomology algebra $H_{orb}^\bullet([V/G], \mathbb{C})$.

Unfortunately, our task is made quite difficult by the nature of the proof. The proof is really quite roundabout, and it relies heavily on earlier results obtained in other papers and/or by other authors. The identification $H_{orb}^\bullet([V/G], \mathbb{C}) \cong H^\bullet(X, \mathbb{C})$ is split into a series of algebra isomorphisms most of which have been known before. Only some isomorphisms in the middle still need proving — and the things isomorphic are really quite far removed from either $H_{orb}^\bullet([V/G])$ or $H^\bullet(X)$. The upshot is that there is very little in [GK] which has to do with actual multiplication in cohomology, be it orbifold cohomology or the usual one. For that matter, more than half of the paper is taken with hardcore deformation theory, and it even has nothing to do with symplectic quotient singularities or crepant resolutions. This makes [GK] hard to read for people not familiar with the field.

Faced with this difficulty, we have adopted a solution which is somewhat radical — we hope that it is justified, if not by pedagogical reasons, then at least by its comparative novelty. There is an alternative prove of [GK, Theorem 1.2] which is shorter, more conceptual and more to the point. However, it needs some intermediate results that are presently not known. Worse than that, some of them might not even be true. Nevertheless, it is this fake “proof” that we will present in this paper. By this, we hope to achieve two goals: firstly, we show the main ideas behind the real proof without bothering with making them work, secondly, we motivate the introduction of some techniques, such as Hochschild cohomology, which *a priori* would seem quite foreign to the subject. Then in the last section, we describe how the actual proof in [GK] works — what additional ingredients one has to add to the brew so that all the “fake” statements of earlier sections become

theorems.

Therefore this paper should very much be thought of as an exposition — very little, if anything, will be actually proved. We hope that at least, it will make a passable read.

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1 Orbifold cohomology algebra

We start with combinatorics. Let V be a finite dimensional complex vector space. Let $g : V \rightarrow V$ be an automorphism of V of some finite order r . Then V can be decomposed into g -eigenspaces, $V = V_1 \oplus \cdots \oplus V_k$, where g acts on V_i by multiplication by $\exp(2\pi\sqrt{-1} \cdot a_i/r)$ for some integer a_i , $0 \leq a_i < r$. Denote by $m_i = \dim V_i$ the multiplicity of the eigenvalue $\exp(2\pi\sqrt{-1} \cdot a_i/r)$. By the *age* of the automorphism g one understands the sum

$$\text{age } g = \sum_{1 \leq i \leq k} m_i \frac{a_i}{r}.$$

The age is *a priori* a rational number. However, the automorphism g acts on $\det V$ by $\exp(2\pi\sqrt{-1} \text{age } g)$. Therefore if g preserves the volume on V , its age is actually an integer. It is obviously invariant under conjugation.

Assume given a finite subgroup $G \subset SL(V)$, and let \overline{G} denote the set of conjugacy classes of elements in G . By the *orbifold cohomology space* $H_{orb}^\bullet([V/G], \mathbb{C})$ we will understand the graded vector space defined by

$$H_{orb}^k([V/G], \mathbb{C}) = \bigoplus_{g \in \overline{G}, k=2 \text{ age } g} \mathbb{C} \cdot g.$$

In other words, $H_{orb}^\bullet([V/G], \mathbb{C})$ is generated by conjugacy classes $g \in \overline{G}$, each placed in degree $2 \text{ age } g$, and the odd part of $H_{orb}^\bullet([V/G], \mathbb{C})$ is trivial. One can also define a bigger space $H_G^\bullet(V, \mathbb{C})$ by

$$H_G^k(V, \mathbb{C}) = \bigoplus_{g \in G, k=2 \text{ age } g} \mathbb{C} \cdot g.$$

In other words, $H_G^\bullet(V, \mathbb{C})$ is just the vector space $\mathbb{C}[G]$ spanned by the group G and graded by twice the age. The group G acts on $H_G^\bullet(V, \mathbb{C})$ by conjugation, and $H_{orb}^\bullet([V/G], \mathbb{C})$ is naturally identified with the subspace of G -invariant vectors in $H_G^\bullet(V, \mathbb{C})$.

To define a multiplication on $H_{orb}^\bullet([V/G], \mathbb{C})$ – we follow the exposition of [CR] given in [FG] – one defines a G -invariant multiplication on the bigger space $H_G^\bullet(V, \mathbb{C})$. This is given in the basis numbered by $g \in G$ by

$$g_1 \cdot g_2 = c(g_1, g_2) g_1 g_2,$$

where $c(g_1, g_2) = 1$ if $\text{age } g_1 + \text{age } g_2 = \text{age}(g_1 g_2)$ and 0 otherwise. It is extremely non-trivial to prove that this really defines an associative multiplication on $H_G^\bullet(V, \mathbb{C})$. Once one has proved this, one descends to the subspace $H_{orb}^\bullet([V/G], \mathbb{C})$ of G -invariant vectors, and observes that the multiplication becomes not only associative, but also commutative.

Assume now that the vector space V is equipped with a non-degenerate skew-symmetric form $\omega \in \Lambda^2 V^*$ and that $G \subset Sp(V)$ preserves ω . Then the orbifold cohomology algebra is much easier to describe.

Lemma 1.1. *For every element $g \in G \subset Sp(V)$, we have*

$$\text{age } g = \frac{1}{2} \text{codim } V^g,$$

where $V^g \subset V$ is the subspace of g -invariant vectors. Moreover, define an increasing filtration $F_\bullet \mathbb{C}[G]$ on the group algebra $\mathbb{C}[G]$ by

$$F_k \mathbb{C}[G] = \bigoplus_{g \in G, \text{codim } V^g \leq k} \mathbb{C} \cdot g.$$

Then this filtration is compatible with the group algebra multiplication, and we have

$$H_G^\bullet(V, \mathbb{C}) \cong \text{gr}^F \mathbb{C}[G].$$

The orbifold cohomology algebra $H_{orb}^\bullet([V/G], \mathbb{C}) = H_G^\bullet(V, \mathbb{C})^G$ is then isomorphic to $\text{gr}^F Z(G)$, where $Z(G) = \mathbb{C}[G]^G$ is the center of the group algebra $\mathbb{C}[G]$.

Proof. Take an element $g \in G$ and consider the associated eigenspace decomposition $V = V_1 \oplus \cdots \oplus V_k$. Since $V \cong V^*$ as a G -module, for every a_i with $0 < a_i < r$, the associated multiplicity m_i must be the same as the multiplicity associated to $r - a_i$. Therefore

$$\text{age } g = \sum_{1 \leq i \leq k} m_i \frac{a_i}{r} = \frac{1}{2} \sum_{1 \leq i \leq k, a_i \neq 0} m_i \left(\frac{a_i}{r} + \frac{r - a_i}{r} \right) = \frac{1}{2} \text{codim } V^g.$$

If we have two elements $g_1, g_2 \in G$, then

$$V^{g_1} \cap V^{g_2} \subset V^{g_1 g_2}.$$

Since for every two vector subspaces $V_1, V_2 \subset V$ we have $\text{codim } V_1 \cap V_2 \leq \text{codim } V_1 + \text{codim } V_2$, this implies that

$$\text{age } g_1 g_2 \leq \text{age } g_1 + \text{age } g_2.$$

This proves that the filtration is indeed multiplicative. The last claim follows. \square

We note that the equality $\text{age } g_1 g_2 = \text{age } g_1 + \text{age } g_2$ is achieved if two things hold: (1) the subspaces V^{g_1} and V^{g_2} intersect transversally, and (2) we have $V^{g_1 g_2} = V^{g_1} \cap V^{g_2}$ (in general, the left-hand side might be strictly bigger). However, (1) is enough.

Lemma 1.2 ([GK, Lemma 1.7]). *Let V be a vector space equipped with a non-degenerate skew-symmetric form ω , and let g_1, g_2 be two symplectic automorphisms of the vector space V . If V^{g_1} and V^{g_2} intersect transversally, then $V^{g_1 g_2} = V^{g_1} \cap V^{g_2}$.*

Proof. We will not need really this claim, but the proof is so simple, we could not resist reproducing it here. It is easy to see that for any symplectic automorphism $g \in Sp(V)$, we have a decomposition $V = V^g \oplus \text{Im}(\text{id} - g)$ which is orthogonal with respect to ω . Now, for any $v \in V^{g_1 g_2}$ we have $g_2 v = g_1^{-1} v$, which implies

$$(g_2 - \text{id})v = (g_1^{-1} - \text{id})v.$$

Both g_1^{-1} and g_2 are symplectic, therefore the left-hand side is orthogonal to V^{g_2} , and the right-hand side is orthogonal to $V^{g_1^{-1}} = V^{g_1}$. Being equal, they both must be orthogonal to $V^{g_1} + V^{g_2} = V$. We conclude that indeed $g_2 v = g_1^{-1} v = 0$. \square

2 Overview of the statements

We will now formulate the main claims of McKay correspondence. Firstly, there is the original cohomological McKay correspondence conjectured in [R1] and proved in [Bat], [DL].

Theorem 2.1 ([Bat], [DL]). *Let V be a vector space, let $G \subset SL(V)$ be a finite subgroup, and let $X \rightarrow V/G$ be a crepant resolution of the quotient V/G . Then there exists a graded vector space isomorphism*

$$H^\bullet(X, \mathbb{Q}) \cong H_{orb}^\bullet([V/G], \mathbb{Q}).$$

The following has also been conjectured in [R1].

Corollary 2.2. *In the assumptions of Theorem 2.1, there exists an isomorphism*

$$K_G^0(V) \cong K^0(X)$$

between the K^0 -group of X and the G -equivariant K^0 -group of V .

To derive this from Theorem 2.1, it suffices to construct a Chern character which would identify $K_G^0(V)$ with $H_{orb}^\bullet([V/G], \mathbb{Q})$; this can be easily done by direct inspection (it is well-known that $K_G^0(V)$ coincides with K_G^0 of a point, and the latter group is just the space of class functions on the group G). Alternatively, one can use the very nice general theory of Chern character and Riemann-Roch Theorem for orbifolds developed by B. Toen in [T].

However, one could also imagine a different proof of Corollary 2.2, and it is for this reason that the statement was given separately in [R1]. Namely, one has the following generalization.

Conjecture 2.3. *In the assumptions of Theorem 2.1, there exists an equivalence*

$$\mathcal{D}_{coh}^b(X) \cong D_G^b(V)$$

between the bounded derived category of coherent sheaves on X and the bounded derived category of G -equivariant coherent sheaves on V .

So far, this remains a conjecture except in the case $\dim V = 2$ (see [KaVa]) and the case $\dim V = 3$, where a very elegant and unexpected proof was given recently in [BKR]. Unfortunately, the methods of [BKR] do not seem to apply in higher dimensions.

Finally, here is [GK, Theorem 1.2].

Theorem 2.4 ([GK, Theorem 1.2]). *In the notation and assumptions of Theorem 2.1, assume in addition that V carries a non-degenerate skew-linear form ω preserved by G . Then there exists a multiplicative isomorphism*

$$(2.1) \quad H^\bullet(X, \mathbb{C}) \cong H_{orb}^\bullet([V/G], \mathbb{C}).$$

As we have noted in the Introduction, this has been proved independently and simultaneously in [V] and [LS] (see also [LQW]) for the case $V = \mathbb{C}^{2n}$, $G = S_n$ is the symmetric group on n letters, $X = \text{Hilb}^{[n]}(\mathbb{C}^2)$ is the Hilbert scheme of subschemes in \mathbb{C}^2 of dimension 0 and length n . W. Wang with co-authors were also able to treat the so-called wreath product case: $V = \mathbb{C}^{2n}$, G is the semidirect product of S_n with the n -fold product Γ^n of a fixed finite subgroup $\Gamma \subset SL(2, \mathbb{C})$, the resolution X is obtained by taking a smooth resolution $X_0 \rightarrow \mathbb{C}^2/\Gamma$ and considering $X = \text{Hilb}^{[n]}(X_0)$.

Note that Theorem 2.4 in fact works for cohomology with coefficients in \mathbb{Q} . In the symplectic case, there is an alternative proof of this theorem [Ka2] which also works over \mathbb{Q} . In fact, in the Hilbert scheme case the multiplicative isomorphism (2.1) holds over \mathbb{Q} and even over \mathbb{Z} (possibly after one makes certain sign changes in the definition of the orbifold cohomology — over \mathbb{C} , any sign changes become irrelevant). The methods of [GK], however, very decidedly work only over \mathbb{C} . It would be interesting to try to use the methods of [Ka2] to check Theorem 2.4 over \mathbb{Q} . However, W. Wang informed me that this might be hopeless — his computations in the wreath product case show that (2.1) probably does not hold for cohomology with coefficients in \mathbb{Q} . Y. Ruan [Ru2] also states his conjecture only over \mathbb{C} .

3 Orbifold cohomology as Hochschild cohomology

Our starting point in proving Theorem 2.4 is the following observation: there is one place in mathematics where the algebra $H_{orb}^\bullet([V/G]) \cong \text{gr}^F Z(G)$ appeared before. This is the computation by M.S. Alvarez [Al] of the Hochschild cohomology of the so-called *cross-product algebra*. Let us explain this point, starting from the definitions. We give very few references in the next two Subsections since the subject is well-represented in the literature; the reader is referred to, for instance, a comprehensive overview given in [L]. For everything related to quantizations, please see [Ko].

3.1 Recollection on Hochschild cohomology Let A be a finitely-generated commutative algebra over \mathbb{C} , and let B be an flat associative algebra over A . Recall that an B -bimodule M is an A -module equipped with two commuting B -module structures, one left and the other right. Equivalently, M is a left $B \otimes_A B^{op}$ -module, where B^{op} denotes the opposite algebra. All B -bimodules form an abelian category. The algebra B itself is tautologically a B -bimodule. For any B -bimodule M , one defines the

Hochschild cohomology groups $HH_A^\bullet(B, M)$ with coefficients in M by

$$HH_A^\bullet(B, M) = \text{Ext}_{B \otimes_A B^{op}}^\bullet(B, M).$$

The Hochschild cohomology groups $HH_A^\bullet(B, B)$ with coefficients in the tautological module B are denoted simply $HH_A^\bullet(B)$ and called the *Hochschild cohomology groups of the algebra B* .

This notion immediately generalizes to the case of sheaves of algebras. Namely, if X is a topological space equipped with a sheaf \mathcal{O} of flat associative A -algebras, then by definition

$$HH_A^\bullet(X) = \text{Ext}_{X \times X^{op}}^\bullet(\mathcal{O}_\Delta, \mathcal{O}_\Delta),$$

where Ext-groups are taken in the category of sheaves of $\mathcal{O} \boxtimes \mathcal{O}^{op}$ -modules on $X \times X$, and \mathcal{O}_Δ is the tautological sheaf supported on the diagonal $\Delta \subset X \times X$. In particular, in this way one defines Hochschild cohomology of flat scheme over $\text{Spec } A$. Even further, one can define Hochschild cohomology of an arbitrary abelian A -linear category, and the resulting groups are *derived Morita-invariant* – this means that if two A -linear abelian categories have equivalent derived categories, then their Hochschild cohomology groups are naturally isomorphic.

Hochschild cohomology $HH_A^\bullet(B)$ carries two natural algebraic structures. The first one is the multiplication given by the Yoneda multiplication in the Ext-groups. The second one is a Lie bracket $\{-, -\}$ of degree -1 called the *Gertsenhaber bracket*. It is not at all obvious from the definition, but it exists nevertheless; moreover, the bracket and the multiplication together form a so-called *Gerstenhaber algebra*. We do not list the axioms here because we will not need them. We only remark that one of them says that the Yoneda multiplication is commutative.

In the case when the algebra B is commutative, finitely generated and smooth over A , the Hochschild cohomology $HH_A^\bullet(B)$ has been computed in the classic paper [HKR]. The answer is

$$HH_A^\bullet(B) = \Lambda_A^\bullet \mathcal{T}(B/A),$$

the exterior algebra generated by the module $\mathcal{T}(B/A)$ of A -linear derivations $\delta : B \rightarrow B$. In other words, Hochschild cohomology classes are the same as polyvector fields on B/A . The multiplication is just the exterior algebra multiplication, and the Gerstenhaber bracket is the so-called *Schouten bracket* of polyvector fields. This result gives rise to a useful modification of the Hochschild cohomology known as *Poisson cohomology*. Recall

that a *Poisson structure* on B over A is given by an A -linear Lie bracket $\{-, -\} : B \otimes_A B \rightarrow B$ which is a derivation with respect to the multiplication in B . Equivalently, a Poisson structure is given by bivector field $\Theta \in \Lambda_A^2 \mathcal{T}(B/A)$ such that $\{\Theta, \Theta\} = 0$ with respect to the Schouten bracket (the Poisson bracket is then $\{f, g\} = \langle \Theta, df \wedge dg \rangle$). In the Hochschild cohomology language, Θ becomes a class in $HH_A^2(B)$ such that $\{\Theta, \Theta\} = 0$ with respect to the Gerstenhaber bracket. Given such a class, one can introduce a map $\delta_\Theta : HH_A^\bullet(B) \rightarrow HH_A^{\bullet+1}(B)$ by setting

$$\delta_\Theta(a) = \{\Theta, a\}.$$

This has been done by J.-L. Brylinski [Br]. It follows from the axioms of the Gerstenhaber algebra, – or equivalently, from the properties of the Schouten bracket, – that $\{\Theta, \Theta\} = 0$ implies $\delta_\Theta^2 = 0$, and that δ_Θ is a derivation with respect to the multiplication in $HH_A^\bullet(B)$. Therefore we obtain a differential-graded algebra $\langle HH_A^\bullet(B), \delta_\Theta \rangle$. Its cohomology groups are denoted by $HP_A^\bullet(B)$ and called the *Poisson cohomology groups* of the Poisson algebra B/A .

We note that this definition makes sense for any B , not only for a smooth and commutative one. If B is not commutative, it no longer makes sense to speak of a Poisson structure on B . Nevertheless, for any $\Theta \in HH_A^2(B)$ with $\{\Theta, \Theta\} = 0$ we do have the differential-graded algebra $\langle HH_A^\bullet(B), \delta_\Theta \rangle$. We will denote its cohomology groups by $HH_\Theta^\bullet(B)$ and call them the *twisted Hochschild cohomology groups*. Note that in the case when B is commutative but not smooth, the notion of a Poisson structure on B is still well-defined. There also exists a natural notion of Poisson cohomology groups $HP_A^\bullet(B)$; however, this notion is more complicated, and the groups themselves in general do *not* coincide with the twisted Hochschild cohomology groups $HH_\Theta^\bullet(B)$ (a large part of [GK, Appendix] is devoted to a detailed investigation of these phenomena).

3.2 Recollection on quantizations. One situation where twisted Hochschild cohomology occurs naturally is the following one. Let B be an associative algebra over \mathbb{C} . Take $A = \mathbb{C}[h]$, the algebra of polynomials in one variable which we denote by h , and let B_h be a flat associative A -algebra such that $B_h/h \cong B$ – in other words, B_h is a one-parameter deformation of the algebra B . Then the deformation theory associates a Hochschild cohomology class $\Theta \in HH_{\mathbb{C}}^2(B)$ to the family B_h . One can compute the Hochschild cohomology $HH_A^\bullet(B_h)$ by using the spectral sequence associated to the h -adic filtration. The term E^1 of this spectral sequence coincides with

$HH_{\mathbb{C}}^{\bullet}(B) \otimes_{\mathbb{C}} A$, and the differential is given by δ_{Θ} (in particular, we have $\{\Theta, \Theta\} = 0$). Thus the term E^2 of the spectral sequence coincides (for dimensional reasons, only modulo h -torsion) with the twisted Hochschild cohomology groups $HH_{\Theta}^{\bullet}(B) \otimes_{\mathbb{C}} A$.

If the algebra B is commutative, then the class Θ induces a Poisson structure on B . The Poisson bracket on B is given by

$$\{f, g\} = \frac{1}{h} (\tilde{f}\tilde{g} - \tilde{g}\tilde{f}) \mod h^2,$$

where \tilde{f}, \tilde{g} are arbitrary elements in B_h with $\tilde{f} = f \mod h$, $\tilde{g} = g \mod h$. The algebra B_h is then said to be a *quantization* of the Poisson algebra B .

We will need a particular case of these general constructions, namely, the case when the commutative algebra B is smooth and *symplectic* – in other words, the skew-symmetric A -bilinear pairing $\Omega^1(B) \otimes_B \Omega^1(B) \rightarrow B$ on the cotangent module $\Omega^1(B)$ given by Poisson bivector $\Theta \in \Lambda^2 \mathcal{T}(B)$ is non-degenerate. In this case, Θ induces an isomorphism $\Omega^1(B) \rightarrow \mathcal{T}(B)$, and one can identify the modules $\Lambda^{\bullet} \mathcal{T}(B)$ of polyvector fields on B with the modules $\Omega^{\bullet}(B)$ of differential forms on $\text{Spec } B$. Under this identification, – and this one of the important results of [Br], – the Poisson differential $\delta_{\Theta} : \Lambda^{\bullet} \mathcal{T}(B) \rightarrow \Lambda^{\bullet+1} \mathcal{T}(B)$ becomes the de Rham differential $d : \Omega^{\bullet}(B) \rightarrow \Omega^{\bullet+1}(B)$. Therefore the Poisson cohomology $HP^{\bullet}(B)$ coincides with the de Rham cohomology $H_{DR}^{\bullet}(\text{Spec } B)$.

The simplest example of a smooth symplectic algebra is the algebra $B = S^{\bullet}(V)$ of polynomials on a symplectic vector space V with symplectic form ω . Then $HP^k(B) = H_{DR}^k(V)$ is \mathbb{C} for $k = 0$ and 0 otherwise. The algebra $S^{\bullet}(V)$ has a standard quantization W_h known as the *Weyl algebra*; it is the associative algebra over $A = \mathbb{C}[h]$ generated by V modulo the relations

$$vw - wv = h\omega(v, w)$$

for all $v, w \in V$. Specializing to $h = 1$, we get the Weyl algebra $W = W_h/(h - 1)$. If $\dim V = 2n$, then W is non-canonically isomorphic to the algebra of differential operators on the affine space \mathbb{A}^n . Since $HP^k(B) = 0$ for $k \geq 1$, the h -adic spectral sequence for $HH_A^{\bullet}(W_h)$ degenerates. We deduce that $HH^0(W) = \mathbb{C}$ and $HH^k(W) = 0$ for $k \geq 1$.

3.3 Computation for the smash-product algebra. For any associative algebra A equipped with an action of a finite group G , introduce the *smash-product algebra* $A \# G$ in the following way: $A \# G = A \otimes_{\mathbb{C}} \mathbb{C}[G]$ as a vector space, and we have

$$(a_1 \cdot g_1)(a_2 \cdot g_2) = a_1(a_2^g) \cdot g_1 g_2$$

for all $a_1, a_2 \in A$ and $g_1, g_2 \in G$. In particular, given a symplectic vector space V and a finite subgroup $G \subset Sp(V)$, we can form the smash-product $W \# G$, where W is the Weyl algebra associated to V . In [Al], M.S. Alvarez has proved the following.

Theorem 3.1 ([Al]). *The Hochschild cohomology algebra $HH^\bullet(W \# G)$ is isomorphic*

$$HH^\bullet(W \# G) \cong \text{gr}^F Z(G)$$

to the associated graded $\text{gr}^F Z(G)$ of the center $Z(G)$ of the group algebra $\mathbb{C}[G]$ with respect to the filtration introduces in Lemma 1.1.

This Theorem is our main motivation for Theorem 2.4. Its proof, of which we will only present a sketch, is based on an earlier result of [AFLS], where $HH^\bullet(W \# G)$ has been computed as a graded vector space. For simplicity, we will first do this computation with W replaced by the commutative polynomial algebra $B = S^\bullet(V)$.

Lemma 3.2. *Consider the smash-product algebra $B \# G$. Then there exist an isomorphism*

$$(3.1) \quad HH^\bullet(B \# G) = \left(\bigoplus_{g \in G} \Omega^\bullet(V^g)[\text{codim } V^g] \right)^G,$$

where for any $g \in G$ we denote by $V^g \subset V$ the subspace of g -invariant vectors in V , and $\Omega^k(V^g)$ is the space of differential forms of degree k on the vector space V^g .

Sketch of the proof. Since the group G is finite, and our base field \mathbb{C} is of characteristic 0, the group G has no cohomology. Therefore we have

$$\begin{aligned} HH^\bullet(B \# G) &= \text{Ext}_{B \# G\text{-bimod}}^\bullet(B \# G, B \# G) \\ &= (\text{Ext}_{B\text{-bimod}}^\bullet(B, B \# G))^G \\ &= (HH^\bullet(B, B \# G))^G \end{aligned}$$

The B -bimodule $B \# G$ can be decomposed

$$B \# G = \bigoplus_{g \in G} B^g,$$

where B^g is B with the standard right B -module structure, and with the left B -module structure twisted by the action of $g \in G$.

Geometrically, finitely-generated B -bimodules are coherent sheaves on $V \times V = \text{Spec}(B \otimes_{\mathbb{C}} B^{op})$; then B^g is just the structure sheaf of the graph $\text{graph } g \subset V \times V$ of the map $g : V \rightarrow V$. In particular, $B = B^{\text{id}}$ itself is the structure sheaf of the diagonal $\Delta \subset V \times V$. To prove the Lemma, it remains to show that

$$HH^{\bullet}(B, B^g) = \Omega^{\bullet}[V^g][\text{codim } V^g].$$

This is immediate. Indeed, $V^g = \Delta \cap \text{graph } g$, and V^g is a symplectic vector space. One computes $HH^{\bullet}(B, B^g)$ by the local-to-global spectral sequence on $V \times V$, and then identifies polyvector fields on V^g with the differential forms by means of the symplectic structure. \square

To deduce Theorem 3.1, one first uses the h -adic spectral sequence to compute $HH^{\bullet}(W \# G)$ as a vector space. At the term E^2 , we have the twisted Hochschild cohomology groups $HH_{\Theta}^{\bullet}(B^g)$. One observes immediately that the differential δ_{Θ} coincides with the de Rham differential on V^g , so that $HH_{\Theta}^{\bullet}(B, B^g) = H_{DR}^{\bullet}(V^g)[\text{codim } V^g]$. The spectral sequence degenerates at E^2 (for instance, because all the non-trivial classes in E^2 have even degrees). We deduce

$$HH^{\bullet}(W \# G) = \left(\bigoplus_{g \in G} \mathbb{C}[\text{codim } V^g] \right)^G,$$

which is exactly (3.1). The multiplicative structure is not easy to see from the spectral sequence. To see it, it is simpler to represent the classes generating $HH^{\bullet}(W \# G)$ by explicit differential forms ω_g (this can be made even more precise and explicit if one uses the Koszul complex for the vector space V to resolve B as a B -module and compute $HH^{\bullet}(B, B^g)$). Then one immediately checks that for every $g, h \in G$, we have $\omega_g \omega_h = \omega_{gh}$ if $V^g, V^h \subset V$ intersect transversally, and $\omega_g \omega_h = 0$ otherwise. By Lemma 1.2, this is exactly the multiplicative structure in $\text{gr}^F \mathbb{Z}(G)$. Note that in fact the same algebra appears as the twisted Hochschild cohomology algebra $HH_{\Theta}^{\bullet}(S^{\bullet}(V) \# G)$.

4 The proofs.

4.1 A proof. We will now present what ought to be the proof of Theorem 2.4. It will only take half a page.

Let V be a symplectic vector space, let $G \subset Sp(V)$ be a finite subgroup, and let $X \rightarrow V/G$ be a crepant resolution of singularities of the

quotient V/G . Since the symplectic form on V is G -invariant, it descends to a symplectic form on the smooth part of the quotient V/G . This form then extends to a closed non-degenerate 2-form on the resolution X (see e.g. [Ka1]). Therefore the variety X is equipped with a natural Poisson structure Θ , and we can compute its de Rham cohomology by comparison with the Poisson cohomology. We then have the isomorphisms

$$(4.1) \quad H_{DR}^\bullet(X) \cong HP^\bullet(X) \cong HH_\Theta^\bullet(X).$$

Now, *assume that Conjecture 2.3 is true*. Note that the category of G -equivariant coherent sheaves on V is equivalent to the category of finitely-generated modules over the smash-product algebra $B \# G = S^\bullet(V) \# G$. Since the Hochschild cohomology is derived Morita-invariant, we have

$$(4.2) \quad H_\Theta^\bullet(X) \cong HH_\Theta^\bullet(B \# G).$$

This finishes the proof: indeed, by Theorem 3.1 (or rather, by its proof) we have $HH_\Theta^\bullet(B \# G) \cong H_{orb}^\bullet([V/G], \mathbb{C})$.

4.2 A debunking. Having given a proof of Theorem 2.4, we will now demolish it.

The most obvious problem with the proof is its reliance on Conjecture 2.3. However, this is not so serious. By a stroke of luck, there exists an approach to this conjecture, which is now under investigation (and the provisory reference for this is [BK]). The author is reasonably certain that in the nearest future Conjecture 2.3 will be proved; hopefully this will happen by the time the present volume is out of print.

Unfortunately, our fake proof of Theorem 2.4 also has many other flaws, and we will now enumerate those.

- (i) We have used isomorphisms (4.1) in spite of the fact that the manifold X is not affine.

This causes two problems. Firstly, the terms in (4.1) might not be defined for non-affine X . As we have explained, the definition of the algebra $HH^\bullet(X)$ does not require X to be affine. To define the twisted version, however, we also need the Gerstenhaber bracket. This is possible to define, too: essentially, the Hochschild cohomology algebra $HH^\bullet(X)$ can be obtained as the hypercohomology of a certain complex of sheaves on X , this complex carries the structure of a Gerstenhaber algebra, the standard homotopy techniques as in e.g. [HS] give a homotopy Gerstenhaber algebra on $HH^\bullet(X)$, and this is equivalent to

a usual Gerstenhaber algebra structure by Kontsevich formality (for this point, we would recommend to consult [H] and references therein). This is really quite roundabout, one certainly would prefer a more direct approach. This is currently the topic of active research. At least one approach has been suggested recently in [RZ], [Ke2].

Secondly, even assuming that all the algebras in (4.1) are well-defined, they might not be isomorphic when X is not affine. And this is exactly what happens, unfortunately: the Hochschild-Kostant-Rosenberg isomorphism $HH^*(X) \cong H^*(X, \wedge^* \mathcal{T}(X))$ holds for non-affine varieties, but *it is not compatible with the Gerstenhaber algebra structure*.

It is probably true that this isomorphism can be *corrected* in a certain precise way so that it becomes multiplicative – this has been claimed without proof in the last part of [Ko], and has been the subject of much research (I do not feel competent enough to provide an exact reference, except for saying that for *compact* X the problem has been definitely completely solved by A. Caldararu [C2]). However, whether this isomorphism is also compatible with the Gerstenhaber bracket is anyone's guess.

- (ii) We have used the derived Morita-invariance of Hochschild homology in (4.2) without any justification.

The derived Morita-invariance property for Hochschild *homology* has been proved several years ago by B. Keller, see [Ke1]. The invariance of cohomology is much simpler. It is so simple that again, I have not been able to track a reference. It might be that no one was diligent enough to write the proof down. For compact X (and derived equivalences of Fourier-Mukai type) everything has been recently written down very carefully by A. Caldararu [C1] (as the foundation for his main results). But again, the question of the Gerstenhaber bracket seems to be open. Quite recently (two weeks ago, at the time of the writing) B. Keller has published a preprint [Ke2] where this question might be solved; unfortunately, my lack of real expertise in the subject forces me to reserve judgement at this time.

- (iii) We have tacitly assumed that the twisting cocycle Θ on $B \# G$ obtained from X is the same as the standard cocycle obtained by descent from the symplectic form on V .

This should be fairly easy to check, *once the Morita-invariance has been solidly established*. Unfortunately, we cannot even start this

checking before we know exactly how the isomorphism (4.2) works.

4.3 How the real proof works. We will now describe, very briefly and sketchily, the main stages of the real proof of Theorem 2.4 given in [GK] (a somewhat more detailed description can be found in [GK, Introduction]). The main idea is the following: all the difficulties appear because we need to work with a non-affine variety X . Things would be much simpler if one could *deform* everything so that the manifold in question becomes affine.

On the resolution side of the picture, a reasonably good deformation theory for smooth non-compact symplectic varieties has been developed in [KaVe]. In particular, it can be applied to a crepant resolution $X \rightarrow V/G$. The result is a smooth family \tilde{X}/B parametrized by the affine space $B = H_{DR}^2(X)$. The approach in [KaVe] only gives a formal deformation; however, there is a natural \mathbb{C}^* -action on V/G and on X which allows one to spread out the deformation to the whole B . The de Rham cohomology $H_{DR}^\bullet(\tilde{X}_b)$ of the fiber \tilde{X}_b is the same for every point $b \in B$. The deformation \tilde{X} also induces a deformation \tilde{Y}/B of the quotient $Y = V/G$, which is affine. The map $X \rightarrow V/G$ extends to a map $\tilde{X} \rightarrow \tilde{Y}$, and the extended map is *one-to-one over a generic point* $b \in B$. Consequently, for generic $b \in B$ the variety $\tilde{X}_b = \tilde{Y}_b$ is simultaneously smooth (being part of \tilde{X}/B) and affine (being part of \tilde{Y}/B).

On the smash-product side, a very nice deformation $H_{t,c}$ of the smash-product algebra $S^\bullet(V) \# G$ has been constructed and studied in some detail in [EG]. The parameter space for this deformation is the product of an affine line with coordinate t and an affine space C of G -invariant functions on the set $S \subset G$ of elements $g \in G$ of age 1 (these elements are also known as *symplectic reflections*). The deformation in the t -direction is entirely non-commutative; in particular, it incorporates the Weyl algebra deformation $W_h \# G$. The deformation $H_{0,c}$, while still non-commutative, is closer to the commutative world – namely, the center Z_c of the algebra $H_{0,c}$ gives a flat deformation of the center $Z_0 \subset S^\bullet(V) \# G$. The latter is obviously just the subalgebra of G -invariant polynomials in $S^\bullet(V)$ – in other words, the algebra of algebraic functions on the quotient V/G .

Moreover, the algebras Z_c carry a natural Poisson structure (which is essentially obtained from the additional deformation in the t -direction), and all of the steps of our fake proof have been solidly established in [EG] *for the affine variety* $\text{Spec } Z_c$. In particular, Conjecture 2.3 becomes a proposition which claims that the algebra $H_{0,c}$ is Morita-equivalent to its center Z_c . Some of the claims – including this Morita-equivalence – require one to

know *a priori* that $\mathrm{Spec} Z_c$ is smooth. If this is known, then the de Rham cohomology $H_{DR}^\bullet(\mathrm{Spec} Z_c)$ has been computed in [EG], and the answer is $H_{orb}^\bullet([V/G], \mathbb{C})$.

These two deformation were the starting point for [GK]. We note that by the additive McKay correspondence, the base spaces $B = H_{DR}^2(X)$ and C are canonically identified. One might hope to identify the deformations $\mathrm{Spec} Z_c$ and \tilde{Y}/B . Then one deduces that $\mathrm{Spec} Z_c$ is smooth for generic $c \in C$, and applies [EG] to compute its cohomology.

Unfortunately, we were not able to prove that such an identification exists, and we had to settle for less. Namely, we prove that \tilde{Y}/B gives a *versal* deformation of the quotient $Y = V/G$ in the class of affine Poisson schemes. In other words, a fiber of any Poisson deformation also occurs as a fiber of the deformation Z_c , but possibly in a non-unique way. In fact, there exists a universal deformation, too, and its base S is also a smooth affine space of dimension $\dim S = \dim B = \dim C$. However, the classifying maps $B \rightarrow S$, $C \rightarrow S$ of the deformations \tilde{Y}/B and Z_c/C are *ramified covers*.

The model for this situation is the case when $\dim V = 2$; then the subgroups $G \subset Sp(V) = SL(2, \mathbb{C})$ are classified by simple Lie algebras \mathfrak{g} of types A , D , E , both the space B and the space C are naturally identified with a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and the space S is the quotient \mathfrak{h}/W of the Cartan algebra \mathfrak{h} by the Weyl group W . We expect that the picture in higher dimension is similar, – in particular, there should exist a natural identification $B = C$. But we are not able to make a precise conjecture at this time.

Be that as it may, what we can prove, eventually, is that every fiber $\mathrm{Spec} Z_c$ of the deformation Z_c occurs as a fiber \tilde{Y}_b of the deformation \tilde{Y} , and b is generic when c is generic. This shows that $\mathrm{Spec} Z_c$ is smooth for generic c and that

$$H_{DR}^\bullet(\mathrm{Spec} Z_c) \cong H_{DR}^\bullet(\tilde{Y}_b) \cong H_{DR}^\bullet(\tilde{X}_b) \cong H_{DR}^\bullet(X).$$

We then invoke [EG] to compute the left-hand side and prove Theorem 2.4.

The only new ingredient in [GK] as compared to [KaVe] and [EG] is a deformation theory for Poisson algebras which is developed far enough to prove the versality of the deformation \tilde{Y}/B .

Remark 4.1. To conclude the paper, we would like to note that the idea to relate orbifold cohomology to something in the Hochschild cohomology world has been introduced by V. Baranovsky [Bar]. He works in full generality – an arbitrary Calabi-Yau quotient V/G – but he used cyclic (or

Hochschild) homology, not Hochschild cohomology. It is certainly a more natural approach. Unfortunately, the homology groups do not carry a multiplicative structure. To try to rectify this, one may attempt to identify Hochschild homology and Hochschild cohomology by cupping a cohomology class with the volume form. However, *this is not what we do*. Our proof also contain a hidden identification $HH_\bullet \cong HH^\bullet$, but the identification uses the symplectic form. On the level of polyvector fields, we use the isomorphism $\mathcal{T}(X) \cong \Omega^1(X)$ – not the isomorphism $\mathcal{T}(X) \cong \Omega^{2\dim X-1}(X)$ obtained by cupping with the volume form.

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